Further results on the uniform observability of discrete-time nonlinear systems

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***Abstract*— In this technical note, the relation between notions of uniform observability of discrete-time nonlinear systems based on injectivity of an observation map (window), the full-rankness of its Jacobian, and a $\mathcal{K}$-function is investigated. It is proved that a system is uniformly observable in the sense of injectivity of the observation map (together with the full-rankness of its Jacobian) if and only if it is so in some $\mathcal{K}$-function senses.**

I. Introduction

Nonlinear observability and observers have been themes of active research for several decades (see, for example, [1]–[24] and references therein). Different nonlinear observers require different "observabilities," hence several different notions related to nonlinear observability have been used in the literature. Among them, some kind of "uniform" observability is especially important in the construction of nonlinear observers of receding horizon type [1], [3], [6], [13], [18], [21].

Three different notions of uniform observability have been widely used in the literature, namely, that based on injectivity of an observation map (or window; a sequence of outputs) as a function of the state, the full-rankness of the Jacobian of the map, and a $\mathcal{K}$-function that determines the relation between the "state error" and corresponding "observation map error," uniformly with respect to all admissible inputs.

The requirement of uniformity with respect to all admissible inputs may appear to be too restrictive; but recently, the author has proved that, for discrete-time nonlinear systems whose state transition and output maps are continuously differentiable, as far as the initial condition and the inputs are on compact sets, uniform observability based on injectivity together with the full-rankness of the observation map is equivalent to its non-uniform counterpart [9]. However, the relation between above mentioned uniform observability and that based on $\mathcal{K}$-functions has been untouched. This technical notes focuses on this subject.

In what follows, several equivalences between uniform observability based on injectivity (or together with the full-rankness) of the observation map and those based on $\mathcal{K}$-functions have been established.

II. Notations and definitions

Consider a discrete-time nonlinear system of the form

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state with $x(0) \in \Omega_X$, $u(t) \in \Omega_U \subset \mathbb{R}^{n_u}$ is the control input, and $y(t) \in \mathbb{R}^{n_y}$ is the output. We assume that Ω_X and Ω_U are compact. Unless explicitly stated otherwise, f is assumed to be C^2 on $\mathbb{R}^n \times \mathcal{N}(\Omega_U)$, where $\mathcal{N}(\Omega_U)$ is an open set containing Ω_U, and h is assumed to be C^2 on $\mathbb{R}^n$.

Observability is the possibility of reconstructing $x(t_0)$ from an output sequence $(y(t_0), y(t_0+1), \ldots)$, but for time-invariant systems, t_0 is immaterial. Therefore, we restrict our attention to the case where $t_0 = 0$.

By $\phi_t(x(0); u)$, we denote the solution of (1) at the time instant t initialized with $x(0)$ at $t = 0$, that is, $\phi_0(x(0); u) = x(0)$, and for $t > 0$, $\phi_t(x(0); u) = f(\phi_{t-1}(x(0); u), u(t-1))$. Similarly, we define a t-length observation map (window) by

$$\eta_t(x(0); u) = \begin{pmatrix} h(x(0)) \\ \cdots \\ h(\phi_{t-1}(x(0); u)) \end{pmatrix}. \tag{2}$$

We denote the finite sequence of inputs $(u(0), \ldots, u(t))$ by $u[0, t]$ and the infinite sequence $(u(t), u(t+1), \ldots)$ by $u[t, \infty]$. The countable product of Ω_U is denoted by $\prod_{\mathbb{N}} \Omega_U$. A sequence $(x_1, x_2, \ldots)$ is denoted by $(x_k)_{k\in\mathbb{N}}$. Note that, in this notation, the subscript has no relation with the time. The symbol $\|\cdot\|$ denotes the Euclidean norm for a vector, and the induced norm for a matrix. The symbol $B(x, r)$ denotes the open ball of radius r centered at x, and S^n denotes the n-dimensional unit sphere centered at the origin. For a bounded set A, $\mathrm{CH}\, A$ denotes the minimum compact and convex set containing A. Product spaces are assumed to be equipped with the product topology, and subspaces are assumed to be equipped with the relative topology.

Typically, uniform observabilities based on injectivity and full-rankness of the observation map are formulated as follows (actually, they are defined in various ways; see [2], [4]–[7], [9], [10], [12], [14], [17], [19], [22], [25], [26] for detail).

Definition 1 [2], [4]–[6], [9], [10], [12], [14], [19], [22], [25], [26] *The system (1) is said to be uniformly observable on Ω_X with respect to all admissible inputs if $\exists N > 0$, $\forall u[0, \infty] \in \prod_{\mathbb{N}} \Omega_U$, the map $\eta_N(x; u)$ is injective as a function of x* *.

Definition 2 [4]–[7], [9], [10], [17], [19] *The system (1) is said to satisfy the uniform observability rank condition on Ω_X with respect to all admissible inputs if $\exists N > 0$, $\forall x \in \Omega_X$, $\forall u[0, \infty] \in \prod_{\mathbb{N}} \Omega_U$, $\operatorname{rank} \left.\frac{\partial \eta_N}{\partial x}\right|_{(x;u)} = n$.*

As for $\mathcal{K}$-functions†, several slightly different "uniform observabilities" based on $\mathcal{K}$-functions have been used in the literature [1], [3], [6], [15], [21]. The following is a typical one.

Definition 3 [3] *The system (1) is said to be $\mathcal{K}$-uniformly observable on Ω_X with respect to all admissible inputs if*

*The function $\eta_N(x; u)$ is affected only by the finite "head" sequence $u[0, N-2]$ and independent of the tail sequence $u[N-1, \infty]$. Similar remarks also hold for subsequent definitions.

†A function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{K}$-function if it is continuous, $\varphi(0) = 0$, and is strictly increasing, where $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. Commonly, the domain of a $\mathcal{K}$-function is the half-open interval $[0, \infty)$ (see, for example, [27], [28]), but in this technical note, we use this notion in a bit wider sense and include the cases where the domain is any half-open or closed interval of the form $[0, a)$ or $[0, a]$, where $a > 0$.

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$\exists N > 0$, $\forall x_1, x_2 \in \Omega_X$, $\forall u[0,\infty] \in \prod_{\mathbb{N}} \Omega_U$,

$$\varphi(\|x_1 - x_2\|) \leq \|\eta_N(x_1; u) - \eta_N(x_2; u)\| \quad (3)$$

for some $\mathcal{K}$-function $\varphi(\cdot)$ ‡.

It is possible to construct an observer through a left inverse of $\eta_N(x;u)$. However, in many cases, the "state estimation error" of an observer is expected to have finite sensitivity against small errors in the "observation map error. " Definition 3 is insufficient to fulfill this requirement, and a stronger condition is required.

Definition 4 [3] *The observation map $\eta_N(x;u)$ is said to have a finite sensitivity to the state* § *if the $\mathcal{K}$-function $\varphi(\cdot)$ of Definition 3 satisfies the following property:*

$$\delta = \inf_{x_1, x_2 \in \Omega_X, x_1 \neq x_2} \frac{\varphi(\|x_1 - x_2\|)}{\|x_1 - x_2\|} > 0. \quad (4)$$

In what follows, the length of the observation map is fixed at N. Therefore, the input sequence that actually affects the observability is $u[0, N-2]$. To avoid the overuse of the tedious notation $u[0, N-2] \in \prod_{N-1} \Omega_U$, we denote it as $\overline{u} \in \overline{\Omega}_U$, and rewrite $\eta_N(x;u)$ as $\eta(x, \overline{u})$ with omitting the subscript N.

III. Main results

We begin with a technical lemma that is required to show the equivalence between the conditions of Definition 1 and Definition 3.

Lemma 5 *Let $I = [0,1]$ or $[0,\infty)$, and $\varphi_0 : I \to \mathbb{R}_{\geq 0}$ be a monotone nondecreasing function with $\varphi_0(0) = 0$ and $\varphi_0(\lambda) \neq 0$ for $\lambda \neq 0$. Then, there is a $\mathcal{K}$-function φ that satisfy $\varphi \leq \varphi_0$ over I. If $I = [0,\infty)$ and φ_0 is unbounded, then φ is a $\mathcal{K}_\infty$-function* ¶.

Proof. First, let $I = [0,1]$, $(a_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence of positive real numbers that converges to zero with $a_0 = 1$, and $(c_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence of positive real numbers that converges to zero with $0 < c_0 < 1$. For each k, define $b_k = c_{k+1}\varphi_0(a_{k+1})$, and let

$$\varphi(\lambda) = \begin{cases} 0, & \lambda = 0, \\ b_{k+1}\frac{a_k - \lambda}{a_k - a_{k+1}} + b_k \frac{\lambda - a_{k+1}}{a_k - a_{k+1}}, & \lambda \in [a_{k+1}, a_k]. \end{cases}$$

Because the sequence $(b_k)_{k\in\mathbb{N}}$ converges to zero, φ is continuous at $\lambda = 0$, and it is obviously continuous at $\lambda \neq 0$. By definition, φ is strictly increasing. We prove that $\varphi(\lambda) \leq \varphi_0(\lambda)$. The assertion is obvious for $\lambda = 0$. If $\lambda \neq 0$, $\exists k$, $\lambda \in [a_{k+1}, a_k]$, and

$$\varphi(\lambda) \leq b_k = c_{k+1}\varphi_0(a_{k+1}) < \varphi_0(a_{k+1}) \leq \varphi_0(\lambda).$$

Hence φ is the desired $\mathcal{K}$-function.

Next, let $I = [0,\infty)$. We assume that, over the interval $[0,1]$, φ has been partially constructed by the above procedure already. By definition,

$$\varphi(a_0) = c_1\varphi_0(a_1). \quad (5)$$

Let $(\alpha_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers that diverge to infinity with $\alpha_0 = a_1$ and $\alpha_1 = a_0$, and $(\gamma_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers that converges to 1 with $\gamma_0 = c_1$ and $\gamma_1 = c_0$. For each $k \geq 1$, define $\beta_k = \gamma_{k-1}\varphi_0(\alpha_{k-1})$, and let $\varphi(\lambda) = \beta_k \frac{\alpha_{k+1} - \lambda}{\alpha_{k+1} - \alpha_k} + \beta_{k+1}\frac{\lambda - \alpha_k}{\alpha_{k+1} - \alpha_k}$ for $\lambda \in [\alpha_k, \alpha_{k+1}]$. Note that, by definition, the sequence $(\beta_k)_{k\in\mathbb{N}\setminus\{0\}}$ is strictly increasing. If $\lambda = \alpha_1 (= a_0)$, then $\varphi(\lambda) = \beta_1 = \gamma_0\varphi_0(\alpha_0) = c_1\varphi_0(a_1)$, which coincides with (5). Hence φ is continuous. By definition, φ is strictly increasing, and $\varphi(\lambda) \leq \varphi_0(\lambda)$ already for $\lambda \in [0,1]$. If $\lambda \in [\alpha_k, \alpha_{k+1}]$ for some $k \geq 1$, then

$$\varphi(\lambda) \leq \beta_{k+1} = \gamma_k\varphi_0(\alpha_k) < \varphi_0(\alpha_k) \leq \varphi_0(\lambda).$$

Hence $\varphi \leq \varphi_0$ on $[0,\infty)$, and φ is the desired $\mathcal{K}$-function.

If $I = [0,\infty]$ and φ_0 is unbounded, $\beta_k \to \infty$ as $k \to \infty$ because $\gamma_k \to 1$ and $\varphi_0(\alpha_k) \to \infty$ as $k \to \infty$. Hence φ is a $\mathcal{K}_\infty$-function. □

Remark 6 In some situations (such as Definition 4), a $\mathcal{K}$-function φ is required to satisfy the condition that $\inf_{\lambda>0}\frac{\varphi(\lambda)}{\lambda} > 0$. It is possible to construct a $\mathcal{K}$-function with this property, as far as $\inf_{\lambda>0}\frac{\varphi_0(\lambda)}{\lambda} > 0$, but the procedure is more elaborate than that given in the proof of Lemma 5.

Because the analysis for the case $I = [0,1]$ is contained in that for the case $I = [0,\infty)$, we consider the latter case only.

Let $(a_k)_{k\in\mathbb{N}}$ and $(\alpha_k)_{k\in\mathbb{N}}$ be as those given in the proof of Lemma 5, with the additional properties that $\inf_{k\in\mathbb{N}}\frac{a_{k+1}}{a_k} > 0$ and $\inf_{k\in\mathbb{N}}\frac{\alpha_k}{\alpha_{k+1}} > 0$ (both of them are fulfilled for, say, geometric sequences.) Let $(c_k)_{k\in\mathbb{N}}$ be a strictly decreasing sequence of positive real numbers with $0 < c_0 < 1$ which converges to a *non-zero* limit. The definitions of $(\gamma_k)_{k\in\mathbb{N}}$, b_k, β_k, and φ are identical to those given in the proof of Lemma 5.

Note that, because φ_0 is nonnegative and monotone, φ_0 has the limit from the right at $\lambda = 0$. We divide the following discussions in two cases, according to the value of $l_\varphi = \lim_{\lambda\to+0}\varphi_0(\lambda)$.

First, consider the case where $l_\varphi = 0$. Even though $(c_k)_{k\in\mathbb{N}}$ converges to a non-zero limit, $\varphi(\lambda) \to 0$ as $\lambda \to 0$ because $\varphi_0(\lambda) \to 0$ as $\lambda_0 \to 0$, hence φ is continuous at the origin. By the analysis given in the proof of Lemma 5, φ is strictly increasing and continuous at $\lambda > 0$. Therefore, φ is a $\mathcal{K}$-function with $\varphi \leq \varphi_0$. On each interval $[a_{k+1}, a_k]$ or $[\alpha_k, \alpha_{k+1}]$, $\varphi(\lambda)$ is a linear function, hence $\frac{\varphi(\lambda)}{\lambda}$ takes the minimum at the border. Therefore, $\inf_{\lambda>0}\frac{\varphi(\lambda)}{\lambda} = \min\{\inf_{k\in\mathbb{N}}\{\frac{\varphi(a_k)}{a_k}\}, \inf_{k\in\mathbb{N}\setminus\{0\}}\{\frac{\varphi(\alpha_k)}{\alpha_k}\}\}$. By definitions of b_k and β_k, $\frac{\varphi(a_k)}{a_k} = c_{k+1}\frac{a_{k+1}}{a_k}\frac{\varphi_0(a_{k+1})}{a_{k+1}}$ and $\frac{\varphi(\alpha_k)}{\alpha_k} = \gamma_{k-1}\frac{\alpha_{k-1}}{\alpha_k}\frac{\varphi_0(\alpha_{k-1})}{\alpha_{k-1}}$. Because $\inf_{k\in\mathbb{N}}\frac{a_{k+1}}{a_k} > 0$ and $\inf_{k\in\mathbb{N}}\frac{\alpha_k}{a_{k+1}} > 0$, $(c_k)_{k\in\mathbb{N}}$ converges to a non-zero limit, $(\gamma_k)_{k\in\mathbb{N}}$ converges to 1 and $\inf_{\lambda>0}\frac{\varphi_0(\lambda)}{\lambda} > 0$, $\inf_{\lambda>0}\frac{\varphi(\lambda)}{\lambda} > 0$, as desired.

‡Some authors (e. g., [3]) define $\mathcal{K}$-uniform observability in the squared form, that is, $\varphi(\|x_1 - x_2\|^2) \leq \|\eta_N(x_1;u) - \eta_N(x_2;u)\|^2$. However, the difference between Definition 3 and this is immaterial, because for a $\mathcal{K}$-function $\varphi(\lambda)$, all of $\varphi(\lambda^2)$, $\varphi(\sqrt{\lambda})$, $\varphi^2(\lambda)$, and $\sqrt{\varphi(\lambda)}$ are $\mathcal{K}$-functions.

§This term does not seem to be standard.

¶A function φ is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and $\varphi(t) \to \infty$ as $t \to \infty$ [27], [28].

Second, consider the case where $l_\varphi > 0$. Let

$$\widetilde{\varphi}_0(\lambda) = \begin{cases} \lambda\varphi_0(\lambda), & \lambda \in [0,1], \\ \varphi_0(\lambda), & \text{otherwise.} \end{cases}$$

Then, $\widetilde{\varphi}_0 \le \varphi_0$. Moreover, for $\lambda \in (0,1]$, $\frac{\widetilde{\varphi}_0(\lambda)}{\lambda} = \varphi_0(\lambda)$, hence $\inf_{\lambda\in(0,1]} \frac{\widetilde{\varphi}_0(\lambda)}{\lambda} = l_\varphi > 0$, and $\inf_{\lambda\in[1,\infty)} \frac{\widetilde{\varphi}_0(\lambda)}{\lambda} = \inf_{\lambda\in[1,\infty)} \frac{\varphi_0(\lambda)}{\lambda} \ge \inf_{\lambda>0} \frac{\varphi_0(\lambda)}{\lambda} > 0$. Moreover, $\lim_{\lambda\to+0} \widetilde{\varphi}_0(\lambda) = 0$. Therefore, applying the construction of the first part of the proof to $\widetilde{\varphi}_0$ instead of φ_0 gives the desired result.

Now, we state and prove the first equivalence.

Theorem 7 *Assume that f and h of (1) are continuous. Then, the system (1) satisfies the condition of Definition 1 if and only if it satisfies the condition of Definition 3.*

Proof. Because the "if" part is obvious, we prove the "only if" part only.

Let $D_\lambda = \{(x_1, x_2, \overline{u}) \in \Omega_X \times \Omega_X \times \overline{\Omega}_U : \|x_1 - x_2\| \ge \lambda\}$, and define

$$\varphi_0(\lambda) = \min_{(x_1,x_2,\overline{u})\in D_\lambda} \|\eta(x_1, \overline{u}) - \eta(x_2, \overline{u})\|.$$

Because $\lambda \le \lambda'$ implies that $D_\lambda \supset D_{\lambda'}$, φ_0 is monotone nondecreasing. Moreover, $\varphi_0(0) = 0$ from construction and $\varphi_0(\lambda) \ne 0$ because $\eta(x, \overline{u})$ is injective on Ω_X for each $\overline{u}$. Hence, by Lemma 5, it is possible to construct a desired $\mathcal{K}$-function φ from φ_0 [‖]. □

Next, assume that the conditions of Definition 1 and Definition 2 are satisfied. Then, the conditions of Definition 3 and Definition 4 are satisfied, and even more.

Proposition 8 *Assume that the conditions of Definition 1 and Definition 2 are fulfilled. Then, $\exists\delta > 0$, $\forall x_1, x_2 \in \Omega_X$, $\forall \overline{u} \in \overline{\Omega}_U$,*

$$\|\eta(x_1, \overline{u}) - \eta(x_2, \overline{u})\| \ge \delta\|x_1 - x_2\|. \tag{6}$$

Proof. Let $J(x, \overline{u}) = \frac{\partial\eta}{\partial x}(x, \overline{u})$. We first prove that $\exists\mu > 0$, $\forall x \in \Omega_X$, $\forall \overline{u} \in \overline{\Omega}_U$, $\forall p \in S^n$,

$$\|J(x, \overline{u})p\| \ge \mu. \tag{7}$$

Because $\|J(x, \overline{u})p\|$ is a continuous function of $(x, \overline{u}, p)$ over the domain $\Omega_X \times \overline{\Omega}_U \times S^n$, and the domain is compact, $\|J(x, \overline{u})p\|$ takes a well-defined minimum μ at some $(x_\mu, \overline{u}_\mu, p_\mu)$. Because $J(x, \overline{u})$ is of full rank, $\mu > 0$. This implies that $\|J(x, \overline{u})p'\| \ge \mu\|p'\|$ for any $p' \in \mathbb{R}^n$.

Next, we prove that $J(x, \overline{u})$ satisfies the Lipschitz condition with respect to x, uniformly for all $\overline{u}$, on $\mathrm{CH}\,\Omega_X \times \overline{\Omega}_U$. Let the l-th row vector of $J(x, \overline{u})$ be $j_l(x, \overline{u})$, and $x, x + p \in \mathrm{CH}\,\Omega_X$. Then,

$$\begin{aligned} j_l^T(x + p, \overline{u}) - j_l^T(x, \overline{u}) &= \int_0^1 \frac{d}{d\lambda} j_l^T(x + \lambda p, \overline{u}) d\lambda \\ &= \int_0^1 \frac{\partial j_l^T}{\partial x}(x + \lambda p, \overline{u}) p d\lambda. \end{aligned} \tag{8}$$

[‖]The function φ_0 may be discontinuous. This happens typically when Ω_X is not connected.

Let $\Gamma(l) = \max_{x\in\mathrm{CH}\,\Omega_X, \overline{u}\in\overline{\Omega}_U} \|\frac{\partial j_l^T}{\partial x}(x, \overline{u})\|$. Then, by (8),

$$\|j_l(x + p, \overline{u}) - j_l(x, \overline{u})\| \le \Gamma(l)\|p\| \int_0^1 d\lambda = \Gamma(l)\|p\|.$$

By letting $\Gamma = \sqrt{N} \max_{l=1,\ldots,N} \Gamma(l)$, we obtain

$$\|J(p + x, u) - J(x, u)\| \le \Gamma\|p\|, \tag{9}$$

as desired.

By using (9), we evaluate the difference between $\eta(x + p, \overline{u}) - \eta(x, \overline{u})$ and $J(x, \overline{u})p$, uniformly with respect to all $\overline{u}$, where $x, x + p \in \mathrm{CH}\,\Omega_X$. Let

$$r(x, p, \lambda, \overline{u}) = J(x + \lambda p, \overline{u}) - J(x, \overline{u}). \tag{10}$$

Then, by (9),

$$\|r(x, p, \lambda, \overline{u})\| \le \Gamma\|\lambda p\|. \tag{11}$$

By evaluating the right hand side of

$$\begin{aligned} \eta(x + p, \overline{u}) - \eta(x, \overline{u}) &= \int_0^1 \frac{d}{d\lambda} \eta(x + \lambda p, \overline{u})\, d\lambda \\ &= \int_0^1 J(x + \lambda p, \overline{u}) p\, d\lambda \end{aligned}$$

with using (10) and (11), we obtain that

$$\eta(x + p, \overline{u}) - \eta(x, \overline{u}) - J(x, \overline{u})p = \int_0^1 r(x, p, \lambda, \overline{u}) p\, d\lambda \tag{12}$$

and

$$\left\| \int_0^1 r(x, p, \lambda, \overline{u}) p\, d\lambda \right\| \le \frac{\Gamma}{2}\|p\|^2. \tag{13}$$

Let $x_1, x_2 \in \Omega_X$. By (7), (12), and (13),

$$\|\eta(x_1, \overline{u}) - \eta(x_2, \overline{u})\| \ge \mu\|x_1 - x_2\| - \frac{\Gamma}{2}\|x_1 - x_2\|^2.$$

Thus, if $\|x_1 - x_2\| \le \frac{\mu}{\Gamma}$,

$$\|\eta(x_1, \overline{u}) - \eta(x_2, \overline{u})\| \ge \frac{\mu}{2}\|x_1 - x_2\|.$$

Next, consider the set $D_{\mu/\Gamma} = \{(x_1, x_2, \overline{u}) \in \Omega_X \times \Omega_X \times \overline{\Omega}_U : \|x_1 - x_2\| \ge \frac{\mu}{\Gamma}\}$. If $D_{\mu/\Gamma} = \emptyset$, we are done by letting $\delta = \frac{\mu}{2}$, hence assume that $D_{\mu/\Gamma} \ne \emptyset$, and let $\delta_0 = \min_{(x_1,x_2,u)\in D_{\mu/\Gamma}} \|\eta(x_1, \overline{u}) - \eta(x_2, \overline{u})\|$. Because $D_{\mu/\Gamma}$ is compact, the minimum is well defined, and it is positive because η is injective for each $\overline{u}$. Then, on the set $D_{\mu/\Gamma}$,

$$\|\eta(x_1, \overline{u}) - \eta(x_2, \overline{u})\| \ge \frac{\delta_0}{\mathrm{diam}\,\Omega_X}\|x_1 - x_2\|,$$

where $\mathrm{diam}\,\Omega_X = \sup_{x_1,x_2\in\Omega_X} \|x_1 - x_2\|$. Thus, by letting $\delta = \min\{\frac{\mu}{2}, \frac{\delta_0}{\mathrm{diam}\,\Omega_X}\}$, we obtain (6). □

It is immediate that $\delta\|x_1 - x_2\|$ given in (6) is a $\mathcal{K}$-function that satisfies the requirements of Definitions 3 and 4.

However, as the following example shows, a system that satisfies the conditions of Definitions 3 and 4 does not always satisfy the condition of Definition 2.

Example 1 Let $\Omega_X = \{(\lambda, 0) : \lambda \in [0,1]\} \cup \{(1, \lambda) : \lambda \in [-1,1]\} \subset \mathbb{R}^2$ and $\eta : \Omega_X \ni (v, w) \mapsto (v, v^2 w) \in \mathbb{R}^2$. Then, η coincides with the identity function on Ω_X, and for (v, w) and (v', w'), the standard Euclidean norm

$\sqrt{(v-v')^2+(w-w')^2}$ serves as a $\mathcal{K}$-function that satisfies the conditions of Definitions 3 and 4. However, the Jacobian of η is singular at the origin.

On the other hand, the conditions of Definition 1 and Definition 2 permit us to enlarge the set in which the inequality (6) is valid to a set of the form $G\times\overline{\Omega}_U$, where G is an open set containing $\Omega_X\times\Omega_X$, and this gives us the desired equivalence to the conditions of Definitions 3 and 4 in a slightly modified form.

To see this, the following elementary lemma is necessary

Lemma 9 ** *Let X, Y be topological spaces. If $A\subset X$, $B\subset Y$, B is compact in Y, $A\times B\subset G$, and G is open in $X\times Y$. then, for some open set G_A of X, $A\subset G_A$ and $G_A\times B\subset G$.*

Proof. Fix some $a\in A$. Then, $\forall b\in B$, $\exists G_A(b)$: open in X, $\exists G_B(b)$: open in Y, $(a,b)\in G_A(b)\times G_B(b)\subset G$. Because $B\subset\cup_{b\in B}G_B(b)$ and B is compact, $\exists l$, $\exists b_1,\dots,b_l$, $B\subset\cup_{k=1}^{l}G_B(b_k)$. Let $G_A(a)=\cap_{k=1}^{l}G_A(b_k)$. Then, $G_A(a)\times B\subset G_A(a)\times\cup_{k=1}^{l}G_B(b_k)\subset G$. By letting $G_A=\cup_{a\in A}G_A(a)$, we obtain the desired open set. □

Theorem 10 *The system (1) satisfies the conditions of Definition 1 and 2 if and only if it satisfies the conditions of Definition 3 and 4 for any $(x_1,x_2,\overline{u})\in G\times\overline{\Omega}_U$ (instead of $\Omega_X\times\Omega_X\times\overline{\Omega}_U$), where G is an open set containing $\Omega_X\times\Omega_X$.*

Proof. Due to the enlargement of $\Omega_X\times\Omega_X$ into G, the "if" part is obvious. Hence, we prove the "only if" part only.

As is already stated, our strategy is to enlarge the set in which the inequality (6) is valid to a set of the form $G\times\overline{\Omega}_U$.

Let $D_0=\Omega_X\times\overline{\Omega}_U\times S^n$. We have already shown that $\forall(x,\overline{u},p)\in D_0$, $\|J(x,\overline{u})p\|\geq\mu$. Let $G_0=\{(x,\overline{u},p)\in\mathbb{R}^n\times\overline{\Omega}_U\times S^n:\|J(x,\overline{u})p\|>\frac{1}{2}\mu\}$. Then, G_0 is open and $D_0\subset G_0$. Hence, by Lemma 9, there is an open set G_A containing Ω_X and $G_A\times\overline{\Omega}_U\times S^n\subset G_0$. By taking the intersection with $B(0,2\max_{x\in\Omega_X}\|x\|)$ if necessary, without loss of generality, one may assume that G_A is bounded. Then, by using the same argument as that of the proof of Proposition 8 (but replacing $\mathrm{CH}\,\Omega_X$ with $\mathrm{CH}\,G_A$), $\exists\Gamma'>0$, $\forall x_1,x_2\in\mathrm{CH}\,G_A$, $\forall\overline{u}\in\overline{\Omega}_U$, $\|J(x_1,\overline{u})-J(x_2,\overline{u})\|\leq\Gamma'\|x_1-x_2\|$, and again by the same argument as that of the proof of Proposition 8, $\forall x_1,x_2\in G_A$, $\forall\overline{u}\in\overline{\Omega}_U$,

$$\|\eta(x_1,\overline{u})-\eta(x_2,\overline{u})\|\geq\frac{1}{2}\mu\|x_1-x_2\|-\frac{1}{2}\Gamma'\|x_1-x_2\|^2.$$

Therefore, if $\|x_1-x_2\|\leq\frac{\mu}{2\Gamma'}$,

$$\|\eta(x_1,\overline{u})-\eta(x_2,\overline{u})\|\geq\frac{1}{4}\mu\|x_1-x_2\|.$$

If the set $\{(x_1,x_2)\in G_A\times G_A:\|x_1-x_2\|\geq\frac{\mu}{2\Gamma'}\}$ is empty, we are done by letting $\delta=\frac{1}{4}\mu$ and $G=G_A\times G_A$. Otherwise, let $G'=\{(x_1,x_2)\in G_A\times G_A:\|x_1-x_2\|<\frac{\mu}{2\Gamma'}\}$, $D_1=\{(x_1,x_2)\in\Omega_X\times\Omega_X:\|x_1-x_2\|\geq\frac{\mu}{2\Gamma'}\}$ and $D_2=D_1\times\overline{\Omega}_U$. Because η is an injection on Ω_X for each $\overline{u}$ and D_2 is compact, $\|\eta(x_1,\overline{u})-\eta(x_2,\overline{u})\|$ takes a non-zero minimum δ_0' on D_2. Let $G_2=\{(x_1,x_2,\overline{u})\in G_A\times G_A\times\overline{\Omega}_U:\|\eta(x_1,\overline{u})-\eta(x_2,\overline{u})\|>\frac{1}{2}\delta_0'\}$. Then, G_2 is open and $D_2\subset G_2$, hence, by Lemma 9, there is an open set G_1 containing D_1 and $G_1\times\overline{\Omega}_U\subset G_2$. Because G_1 is a subset of $G_A\times G_A$, by letting $\delta=\min\{\frac{1}{4}\mu,\frac{\delta_0'}{2\mathrm{diam}\,G_A}\}$ and $G=G'\cup G_1$, (6) holds for all $(x_1,x_2,\overline{u})\in G\times\overline{\Omega}_U$.

Now, the conditions of Definition 3 and 4 are satisfied on $G\times\overline{\Omega}_U$ by the function $\delta\|x_1-x_2\|$. □

Theorem 10 also shows that a $\mathcal{K}$-function defining $\mathcal{K}$-uniform observability may be replaced with a norm function.

Remark 11 It may appear that the distinction between the conditions of Definition 3 and 4 and those given in Theorem 10 is immaterial if the initial condition set Ω_X is open (and is not compact). However, without compactness, Theorem 10 does not hold. Obviously, if Ω_X is open and the conditions of Definition 3 and 4 are fulfilled, then the conditions of Definition 1 and 2 are fulfilled. However, the converse is false. For example, consider the one-dimensional map

$$\eta:\mathbb{R}\ni\lambda\mapsto\arctan\lambda\in\mathbb{R}.$$

This is an injection and $\frac{d\eta}{d\lambda}=\frac{1}{1+\lambda^2}>0$. However, in order for a nonnegative, monotone nondecreasing, and continuous function φ to satisfy $\varphi(|\lambda_1-\lambda_2|)\leq|\arctan\lambda_1-\arctan\lambda_2|$, it should be identically zero.

Remark 12 Among several definitions of observability of discrete-time nonlinear systems, the most convenient one, especially for observers of receding horizon type, is that based on a $\mathcal{K}$-function. Superficially, $\mathcal{K}$-function-based observability (Definition 3 together with Definition 4) looks a bit artificial, but Theorem 10 and the results of [9] shows that this appearance is deceptive. By Theorem 10, they are equivalent to the uniform observability in the sense of injectivity of the observation map together with the full-rankness of its Jacobian, and it is shown in [9] that they are equivalent to their non-uniform counterparts, as far as the initial condition and the inputs are on compact sets. In this sense, $\mathcal{K}$-function-based observability is a natural property.

IV. Conclusion

In this technical note, we have shown that, under C^0 condition, a discrete-time nonlinear system is uniformly observable in the sense of injectivity of the observation map if and only if it is so in $\mathcal{K}$-function sense, and under C^2 condition, it is uniformly observable in the sense of injectivity together with the full rankness of the Jacobian of the observation map if and only if it is so in "finite sensitivity" $\mathcal{K}$-function sense.

It is not known that the latter equivalence holds under weaker conditions such as C^1. For continuous-time systems, several results related to uniform observability have been already obtained [6], but to the best of the author's knowledge, the relation between $\mathcal{K}$-function-based notions and others has not been fully understood yet. To investigate them is left to further research.

**Lemma 9 is almost obvious, and the author does not claim any novelty on this. However, for the sake of completeness, a full proof is given.

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