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Robust nonlinear model predictive control with variable block length

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***Abstract*— Robust model predictive controllers for discrete-time nonlinear systems are proposed in this paper. The algorithms first search for an open-loop controller with some block length, and then try to improve it in a closed-loop fashion by solving minimax problems on-line. It is proved that the controllers are capable of making a subsequence of the state converge into a target set in the presence of bounded disturbances.**

I. Introduction and problem statement

The importance of constructing a robust model predictive controller for a nonlinear system is widely recognized [7], and extensive researches have been carried out [2]–[4], [6], [8]–[10]. However, there is a common drawback in existing methods that many restrictive conditions on the stage cost, terminal constraints, and feasibility are required. This impairs the benefit of model predictive control (MPC) of being readily applicable to a variety of systems. The objective of this paper is to remedy this situation and construct robust model predictive controllers under relatively mild conditions.

Consider a discrete-time nonlinear system of the form

$$x_{k+1} = f(x_k, u_k, w_k), \quad x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^{n_u}, w_k \in \mathbb{R}^{n_w}, \tag{1}$$

where x_k is the state, u_k is the control input, and w_k is the disturbance at the time instant k. We assume that f is C^1. The design objective is to drive the state of the system into a neighborhood of a target point x^{target}. We intend to employ dual-mode MPC, and we assume that once the state is inside the neighborhood, the model predictive controller is turned off and some other controller (not discussed in this paper) is used. Typical applications the author has in mind are i) stabilization and ii) reaching control to a (controlled) attractors. For the former, x^{target} is generally an equilibrium, whereas for the latter, x^{target} is in a neighborhood of the attractor and not always an equilibrium; however, for the simplicity of notation, we let $x^{\text{target}} = 0$.

In what follows, $u_{[i,j]}$ denotes the sequence $(u_i, \ldots, u_j)$. We identify $u_{[i,i]}$ with u_i, and $u_{[i,j]}$ with the empty sequence if $j < i$. The sequence obtained by joining $(u_i, \ldots, u_j)$ and $(u_{j+1}, \ldots, u_l)$ is denoted by $u_{[i,j]} : u_{[j+1,l]}$. Similar notation is used for w_i. The solution of (1) at time N with inputs $u_{[0,N-1]}$ and disturbances $w_{[0,N-1]}$ initialized at x_0 is denoted by $\phi(N, u_{[0,N-1]}, w_{[0,N-1]}, x_0)$; if the disturbances are identically zero, the solution is denoted by $\phi(N, u_{[0,N-1]}, 0, x_0)$. For a vector, $\|\cdot\|$ denotes any norm; for a matrix, it denotes the induced norm. The symbols $B(x, \sigma_x)$ and $\overline{B}(x, \sigma_x)$ denote the open and closed balls centered at x with radius σ_x. For a sequence $u_{[i,j]}$, $\overline{B}^{\infty}(u_{[i,j]}, \sigma_u)$ denotes the set $\{v_{[i,j]} : \forall k \in \{i, \ldots, j\}, v_k \in \overline{B}(u_k, \sigma_u)\}$. The symbol 0 either denotes the scalar zero, or a zero vector of any dimension.

We assume that the disturbances are bounded, that is, for some $\sigma_w > 0$ and for all k, $\|w_k\| \leq \sigma_w$.

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II. Variable block-length controllers

We assume a particular kind of controllability to the target point (the origin) as follows.

Assumption 1 *For any x, there exist a $N \in \mathbb{N}$ and a sequence $u_{[0,N-1]}$ that satisfy:*
i) $\phi(N, u_{[0,N-1]}, 0, x) = 0$, and
ii) $\operatorname{rank}(\partial\phi/\partial u_{[0,N-1]}) = n$ at $(u_{[0,N-1]}, 0, x)$,
where $\partial\phi/\partial u_{[0,N-1]}$ denotes

$$((\partial\phi/\partial u_0)^T, \ldots, (\partial\phi/\partial u_{N-1})^T)^T.$$

The first condition of Assumption 1 is the reachability and the second condition implies that the reachable set contains a non-empty open set.

Assumption 1 does not say anything about the bound of N or the norm of $u_{[0,N-1]}$; however, the following lemma asserts that they are in fact bounded if the initial condition is on a compact set.

Lemma 2 *For any compact set Ω, there exist a $\overline{N} \in \mathbb{N}$ and a continuous function ρ_u on Ω that have the following properties: $(\forall x \in \Omega)(\exists N \leq \overline{N})(\exists u_{[0,N-1]} \in \overline{B}^{\infty}(0, \rho_u(x)))(\phi(N, u_{[0,N-1]}, 0, x) = 0)$.*

Proof. For a $x_0 \in \Omega$, choose some N_{x_0} and $u^0_{[0,N_{x_0}-1]}$ that satisfy Assumption 1. Let p be a vector consisting of elements of $u_{[0,N_{x_0}-1]}$ that satisfy $\operatorname{rank}(\partial\phi/\partial p)(N_{x_0}, u^0_{[0,N_{x_0}-1]}, 0, x_0) = n$, and q be the vector consisting of remaining elements. The vectors corresponding to $u^0_{[0,N_{x_0}-1]}$ are denoted by p_0 and q_0. By omitting unnecessary arguments, we rewrite $\phi(N_{x_0}, u_{[0,N_{x_0}-1]}, 0, x) = \psi(x, q, p)$. Choose some $\varepsilon_v > 0$ and $\varepsilon_p > 0$. Define $\Psi(v, p) = (v^T, (\psi(x_0 + v, q_0, p))^T)^T$, where $v \in \mathbb{R}^n$. Because f is C^1 and the Jacobian of Ψ is nonsingular at $(0, p_0)$, for sufficiently small ε_v and ε_p, Ψ is a homeomorphism on $B(0, \varepsilon_v) \times B(p_0, \varepsilon_p)$.

Because $\psi(x_0, q_0, p_0) = 0$, for sufficiently small $\varepsilon'_v \leq \varepsilon_v$, the set $\{(v^T, x^T)^T : \|v\| \leq \varepsilon'_v, x = 0\}$ is contained in the image of Ψ. This implies that $(\forall x \in \overline{B}(x_0, \varepsilon'_v))(\exists p_x \in$

$B(p_0,\varepsilon_p))(\psi(x,q_0,p_x)=0)$. Let the sequence corresponding to p_x and q_0 be $u^x_{[0,N_{x_0}-1]}$, $\kappa_0 = \max_{j\in\{0,\dots,N_{x_0}-1\}}\|u^0_j\|$, and $\kappa_{x_0}=\kappa_0+\varepsilon_p$. Our conclusion at this stage is as follows:

$$(\forall x\in\overline{B}(x_0,\varepsilon'_v))(\exists u^x_{[0,N_{x_0}-1]}\in\overline{B}^\infty(0,\kappa_{x_0}))$$
$$(\phi(N_{x_0},u^x_{[0,N_{x_0}-1]},0,x)=0).$$

Choose an open set V that contains $\overline{B}(x_0,\varepsilon'_v)$. By Urysohn's lemma, there is a continuous function $\zeta_{x_0}:\mathbb{R}^n\to[0,\kappa_{x_0}+\varepsilon_p]$ that is identically $\kappa_{x_0}+\varepsilon_p$ on $\overline{B}(x_0,\varepsilon'_v)$ and identically zero on the complement of V. Consider the set of triples $\{(N_x,\zeta_x,B(x,\varepsilon_x)) : x\in\Omega\}$, where N_x, ζ_x, and $B(x,\varepsilon_x)$ are constructed for each x by the procedure described above. Because $\{B(x,\varepsilon_x):x\in\Omega\}$ covers Ω and Ω is compact, there is a finite subcollection $\{B(x_j,\varepsilon_{x_j}) : x_j\in\Omega, j\in\{1,\dots,M\}\}$ for some M that also covers Ω. Let $\overline{N}=\max_{j\in\{1,\dots,M\}}N_{x_j}$ and $\rho_u(x)=\sup_{j\in\{1,\dots,M\}}\zeta_{x_j}(x)$. They satisfy the desired property. □

Remark 3 The bound $\rho_u(x)$ obtained in Lemma 2 is of theoretical nature. It is difficult to evaluate it effectively, and there is no assurance that it is compatible to the physical limitations of engineering systems.

Generally, solving the nonlinear equation $\phi(N,u_{[0,N-1]},0,x)=0$ numerically requires infinitely many iterations. Thus, we replace it with the problem of finding $u_{[0,N-1]}$ (an open-loop control sequence) which makes $\|\phi(N,u_{[0,N-1]},0,x)\|\le\alpha\|x\|$ for some $0<\alpha<1$ and $N\le\overline{N}$, whose existence is assured by Lemma 2.

Because our objective is to construct a robust controller, we analyze the effect of $w_{[0,N-1]}$ to $\phi(N,u_{[0,N-1]},0,x)$. Temporally, let us denote $U=(u_0^T,\dots,u_{N-1}^T)^T$, $W=(w_0^T,\dots,w_{N-1}^T)^T$, $\|W\|=(\sum_{k=0}^{N-1}\|w_i\|^2)^{1/2}$, and $F^N(U,W,x)=\phi(N,u_{[0,N-1]},w_{[0,N-1]},x)$. Note that F^N is C^1. Let $\theta\in\prod^n[0,1]\subset\mathbb{R}^n$, and define

$$J^N_w(U,W,\theta,x)=\begin{pmatrix}(\partial F^N_1/\partial W)(U,\theta_1 W,x)\\ \dots\\ (\partial F^N_n/\partial W)(U,\theta_n W,x)\end{pmatrix}. \tag{2}$$

Then, from the intermediate function theorem, for some θ,

$$F^N(U,W,x)-F^N(U,0,x)=J^N_w(U,W,\theta,x)W. \tag{3}$$

Choose a $\sigma_x>0$. Because the set Ω of Lemma 2 is arbitrary, one may assume that $\overline{B}(0,\sigma_x)\subset\Omega$. Let $\sigma_u=\max_{x\in\overline{B}(0,\sigma_x)}\rho_u(x)$,

$$\sigma_{J_w}(N)=\max_{\substack{x\in\overline{B}(0,\sigma_x),\\ U\in\overline{B}^\infty(0,\sigma_u),\\ W\in\overline{B}^\infty(0,\sigma_w),\\ \theta\in\prod^n[0,1]}}\|J^N_w(U,W,\theta,x)\|,$$

and $\sigma_{J_w}=\max_{\{j=1,\dots,\overline{N}\}}\sigma_{J_w}(N)$. Then, by (3), for all $x\in\overline{B}(0,\sigma_x)$,

$$\|F^N(U,W,x)-F^N(U,0,x)\|\le\sigma_{J_w}\sqrt{\overline{N}}\sigma_w. \tag{4}$$

By using above results, we construct a "piecewise open-loop" controller, which is similar to the block MPC of Sun et al. [14]. It consists of open-loop "blocks" with length N_j.

Choose some $\alpha\in(0,1)$. For $x_0\in\overline{B}(0,\sigma_x)$, let $u^0_{[0,N_0-1]}$ be a sequence of inputs that satisfy $\phi(N_0,u^0_{[0,N-1]},0,x_0)\le\alpha\|x_0\|$ with $N_0\le\overline{N}$. Let U_0 be the vector corresponding to $u^0_{[0,N_0-1]}$. Then, from (4), we obtain

$$\|x_{N_0}\|\le\alpha\|x_0\|+\sigma_{J_w}\sqrt{\overline{N}}\sigma_w. \tag{5}$$

Let $M_j=\sum_{i=0}^{j-1}N_i$ (the empty sum implies zero). Inductively, assume that $x_{M_j}\in\overline{B}(0,\sigma_x)$. Then, it is possible to find $N_j\le\overline{N}$ and U_j that satisfy $\|F^{N_j}(U_j,0,x_{M_j})\|\le\alpha\|x_{M_j}\|$. Therefore, for all j,

$$\|x_{M_j}\|\le\alpha^j\|x_0\|+((1-\alpha^j)/(1-\alpha))\sigma_{J_w}\sqrt{\overline{N}}\sigma_w,$$

provided that they are well defined. Therefore, in order for all x_{M_j} to be in $\overline{B}(0,\sigma_x)$, it is sufficient that

$$\|x_0\|+\frac{\sigma_{J_w}\sqrt{\overline{N}}\sigma_w}{1-\alpha}\le\sigma_x, \tag{6}$$

and the above discussion shows that, if (6) is satisfied, the subsequence $(x_{M_j})_{j\in\mathbb{N}}$ converge into the set

$$\overline{B}(0,\sigma_{J_w}\sqrt{\overline{N}}\sigma_w/(1-\alpha)). \tag{7}$$

Remark 4 It should be noted that (6) imposes a restriction on not only the initial condition but also the disturbances, that is,

$$\sigma_w\le(1-\alpha)\sigma_x/(\sqrt{\overline{N}}\sigma_{J_w}) \tag{8}$$

If (8) is not satisfied, no initial condition is permissible. For linear systems, this is not restrictive because σ_{J_w} is a constant and σ_x may be arbitrarily large. For nonlinear systems, σ_{J_w} depends on σ_x; hence, the restriction (8) may become more conservative by making σ_x larger. However, the convergence is "semi-global" in some sense, in that, an arbitrarily large initial condition is permitted by making σ_x larger at the expense of smaller permissible disturbances.

Our new model predictive controllers are the closed-loop counterparts of aforementioned piecewise open loop controller. The endpoint of each block is fixed and the open-loop control sequence inside the block is improved by solving minimax problems on-line.

We first consider the terminal condition only.

Algorithm 1

1) Choose a constant α satisfying $0<\alpha<1$, and let $j=0$ and $M_j=0$.
2) The control terminates if x_{M_j} is inside the target set*. Otherwise, find N_j $(1\le N_j\le\overline{N})$ and $u^0_{[0,N_j-1]}\in\overline{B}^\infty(0,\sigma_u)$ that give
$$\|\phi(N_j,u^0_{[0,N_j-1]},0,x_{M_j})\|\le\alpha\|x_{M_j}\|. \tag{9}$$
Let $k=0$ and proceed to step 3.
3) The control terminates if x_{M_j+k} is inside the target set. If $k=N_j$, let $M_{j+1}=M_j+N_j$,

*We define the target set to be a neighborhood of the target point x^{target} specified by the designer. The controller is supposed to switch to the auxiliary control law after the state reaches into the target set according to the dual-mode policy, but the auxiliary control law itself is not dealt with in this paper.

$j = j+1$, and return to step 2. Otherwise, solve $\min_{u^k_{[k,N_j-1]}\in\overline{B}^\infty(0,\sigma_u)} \max_{w^k_{[k,N_j-1]}\in\overline{B}^\infty(0,\sigma_w)} \|\phi(N_j - k, u^k_{[k,N_j-1]}, w^k_{[k,N_j-1]}, x_{M_j+k})\|$. Let the solution be $u^{k*}_{[k,N_j-1]}$ and $w^{k*}_{[k,N_j-1]}$. Apply u^{k*}_k to (1) to obtain x_{M_j+k+1}; let $k = k+1$, and return to the beginning of this step.

In order to analyze Algorithm 1, we require a simple technical lemma.

Lemma 5 *For a scalar function* $h(x_1,x_2,y_1,y_2)$, *let* $(x_1^b,x_2^b,y_1^b,y_2^b)$ *be a solution of*

$$\min_{x_1,x_2}\max_{y_1,y_2} h(x_1,x_2,y_1,y_2).$$

For x_2^b *and some* y_2^c, *let* (x_1^a,y_1^a) *be a solution of*

$$\min_{x_1}\max_{y_1} h(x_1,x_2^b,y_1,y_2^c).$$

Then, $h(x_1^a,x_2^b,y_1^a,y_2^c)\le h(x_1^b,x_2^b,y_1^b,y_2^b)$.

Proof. $(\forall y_1)(\forall y_2)(h(x_1^b,x_2^b,y_1,y_2)\le h(x_1^b,x_2^b,y_1^b,y_2^b))$ and $h(x_1^a,x_2^b,y_1^a,y_2^c)\le h(x_1^b,x_2^b,y_1^a,y_2^c)$. □

Proposition 6 *Assume that (6) is satisfied. Then, by using Algorithm 1, the subsequence* $(x_{M_j})_{j\in\mathbb{N}}$ *converges into the set (7).*

Proof. From the construction of step 2 of Algorithm 1 and (4), $\|\phi(N_0,u^{0*}_{[0,N_0-1]},w^{0*}_{[0,N_0-1]},x_0)\| \le \|\phi(N_0,u^{0}_{[0,N_0-1]},w^{0*}_{[0,N_0-1]},x_0)\| \le \alpha\|x_0\| + \sigma_{J_w}\sqrt{N}\sigma_w$. By Lemma 5, for all k,

$$\begin{aligned}&\|\phi(N_0-k-1,u^{k+1\,*}_{[k+1,N_0-1]},w^{k+1\,*}_{[k+1,N_0-1]},x_{k+1})\|\\ &=\|\phi(N_0-k-1,u^{\mathrm{act}}_k : u^{k+1\,*}_{[k+1,N_0-1]},w^{\mathrm{act}}_k : w^{k+1\,*}_{[k+1,N_0-1]},x_k)\|\\ &\le\|\phi(N_0-k,u^{k*}_{[k,N_0-1]},w^{k*}_{k,[N_0-1]},x_k)\|,\end{aligned}$$

where u^{act}_k and w^{act}_k are the actual control input and disturbance applied to the plant at the time instant k (note that $u^{\mathrm{act}}_k = u^{k*}_k$). Moreover, $\|x_{N_0}\| \le \|\phi(1,u^{N_0-1\,*}_{N_0-1},w^{N_0-1\,*}_{N_0-1},x_{N_0-1})\|$ because $w^{N_0-1\,*}_{N_0-1}$ is the worst disturbance. Therefore, $x_{M_1}=x_{N_0}$ satisfies (5). Inductively, assume that $\|x_{M_j}\|\le\alpha^j\|x_0\|+(\sum_{i=0}^{j-1}\alpha^i)\sigma_{J_w}\sqrt{N}\sigma_w$. Then, $x_{M_j}\in\overline{B}(0,\sigma_x)$, and arguments similar to the above give $\|x_{M_{j+1}}\|\le\alpha^{j+1}\|x_0\|+(\sum_{i=0}^{j}\alpha^i)\sigma_{J_w}\sqrt{N}\sigma_w$, hence $x_{M_{j+1}}\in\overline{B}(0,\sigma_x)$. Therefore, Algorithm 1 is executable for all j and the subsequence $(x_{M_j})_{j\in\mathbb{N}}$ converges into the set $\overline{B}(0,\sigma_{Jw}\sqrt{N}\sigma_w/(1-\alpha))$. □

If the target set contains the set (7) and is open, Algorithm 1 terminates after finitely many iterations.

Remark 7 The difficult step of Algorithm 1 is to find a N_j and a $u_{[0,N_j-1]}$. Computationally, this step is performed by minimizing $\|\phi(N_j,u^0_{[0,N_j-1]},0,x_{M_j})\|$ with increasing N_j until $\|\phi(N_j,u^0_{[0,N_j-1]},0,x_{M_j})\|\le\alpha\|x_{M_j}\|$ is satisfied (true minimum is unnecessary). The solution can be found relatively easily if one chooses $\alpha\simeq 1$.

Next, we construct a variant of Algorithm 1 that takes the stage cost into account. In the j-th block with length N_j, consider the cost functions

$$\begin{aligned}J_j^k = \sum_{i=0}^{N_j-1} l(i,\phi(i+1,u^{\mathrm{act}}_{[0,k-1]} : u^k_{[k,i]},\\ w^{\mathrm{act}}_{[0,k-1]} : w^k_{[k,i]},x_{M_j}),u_i),\end{aligned}\tag{10}$$

for $k\in\{0,\dots,N_j-1\}$, in which $u^{\mathrm{act}}_{[0,k-1]}$ and $w^{\mathrm{act}}_{[0,k-1]}$ denote the sequences of control inputs and disturbances, respectively, that have already been applied to the system at the time instant M_j+k. In the summation of (10), if $i<k$, $u^{\mathrm{act}}_{[0,k-1]}$ and $w^{\mathrm{act}}_{[0,k-1]}$ are interpreted as $u^{\mathrm{act}}_{[0,i]}$ and $w^{\mathrm{act}}_{[0,i]}$, respectively; u_i is interpreted as an element of $u^{\mathrm{act}}_{[0,k-1]}$ for $i<k$ and an element of $u^k_{[k,\dots,N_j-1]}$ for $i\ge k$. It should be noted that, for evaluating (10), $w^{\mathrm{act}}_{[0,i-1]}$ $(i\le k)$ are in fact unnecessary, because $x_{M_j+i}=\phi(i,u^{\mathrm{act}}_{[0,i-1]},w^{\mathrm{act}}_{[0,i-1]},x_{M_j})$ is known.

Algorithm 2

1) Choose a constant α satisfying $0<\alpha<1$, and let $j=0$ and $M_j=0$.
2) The control terminates if x_{M_j} is inside the target set. Otherwise, find a $N_j\le\overline{N}$ and a $u^0_{[0,N_j-1]}\in\overline{B}^\infty(0,\sigma_u)$ that satisfy (9). Let $k=0$,

$$\begin{aligned}J_j^{-1*} = \max_{w^0_{[0,N_j-1]}\in\overline{B}^\infty(0,\sigma_w)} \|\phi(N_j,u^0_{[0,N_j-1]},\\ w^0_{[0,N_j-1]},x_{M_j})\|,\end{aligned}\tag{11}$$

and proceed to step 3.
3) The control terminates if x_{M_j+k} is inside the target set. If $k=N_j$, let $M_{j+1}=M_j+N_j$, $j=j+1$, and return to step 2. Otherwise, solve $\min_{u^k_{[k,N_j-1]}\in\overline{B}^\infty(0,\sigma_u)}\max_{w^k_{[k,N_j-1]}\in\overline{B}^\infty(0,\sigma_w)} J_j^k$ subject to

$$\begin{aligned}\max_{w^k_{[k,N_j-1]}\in\overline{B}^\infty(0,\sigma_w)} \|\phi(N_j-k,u^k_{[k,N_j-1]},\\ w^k_{[k,N_j-1]},x_{M_j+k})\|\le J_j^{-1*}.\end{aligned}\tag{12}$$

Let the solution be $(u^{k*}_{[k,N_j-1]},w^{k*}_{[k,N_j-1]})$. Apply u^{k*}_k to (1) to obtain x_{M_j+k+1}, let $k=k+1$, and return to the beginning of this step.

Because of (12), Algorithm 2 is executable for all j, and the same convergence property as Proposition 6 is obtained. Let the optimal value of J_j^k be J_j^{k*}. Then, by Lemma 5 (with a slight modification), it follows that $J_j^{0*}\ge J_j^{1*}\ge\cdots$.

Remark 8 Our stage cost has nothing to do with the convergence and may be designed rather freely. Time-varying or even discontinuous stage costs are permitted. A barrier-like function may be used in order to mildly constrain the state inside a set. Exactly constraining the state in a specified set would require an extended function that takes the value $+\infty$ outside the set, but we do not consider such a function here, because introducing such a function causes a problem on feasibility. If it is desirable, instead of (10), a standard receding horizon type stage cost may be used together with the constraint (12).

In this case, the constraint is not "terminal", but the analysis of the convergence is still unaffected.

Thus far, we have fixed the block length N_j after determining N_j and $u_{[0,N_j-1]}$ that satisfy (9). However, a shorter block length is sometimes preferable. A variant of Algorithm 1 that reduces the block length is as follows.

Algorithm 3

1) Choose a constant α satisfying $0<\alpha<1$, and let $j=0$ and $M_j=0$.
2) The control terminates if x_{M_j} is inside the target set. Otherwise, find a $N_j \le \overline{N}$ and a $u^0_{[0,N_j-1]} \in \overline{B}^{\infty}(0,\sigma_u)$ that satisfy (9). Let $k=0$, obtain J_j^{-1*} by (11), let $J_j^* = J_j^{-1*}$, and proceed to step 3.
3) The control terminates if x_{M_j+k} is inside the target set. If $k=N_j$, let $M_{j+1}=M_j+N_j$, $j=j+1$, and return to step 2. Otherwise, for $m=0,\dots,N_j-k-1$, solve
$$\min_{u^k_{[k,k+m]}\in\overline{B}^{\infty}(0,\sigma_u)}\ \max_{w^k_{[k,k+m]}\in\overline{B}^{\infty}(0,\sigma_w)} \|\phi(m+1,u^k_{[k,k+m]},w^k_{[k,k+m]},x_{M_j+k})\|.$$
Let the solution be $(u^{k,m*}_{[k,k+m]}, w^{k,m*}_{[k,k+m]})$, and the corresponding cost be $J^*_{k,m}$. If $J^*_{k,m} \le J^*_j$ for some m, let $J^*_j = J^*_{k,m}$, $N_j = k+m+1$, apply $u_k^{k,m*}$ to (1) to obtain x_{M_j+k+1}, let $k=k+1$, and return to the beginning of this step.

For $m=N_j-k-1$, $J^*_{k,N_j-k-1} \le J^*_j$ is always satisfied. Hence, Algorithm 3 is executable. Its convergence property is the same as that of Algorithm 1.

Remark 9 Our algorithms, which we have been describing as a robust counterparts of the block MPC [14] thus far, may also be interpreted as a non-linear analogue of the tube MPC [8], [12], [13] with more freedom on the selection of the horizon. Our decrease condition (9) may be interpreted as a variant of the constraint tightening which is frequently used in existing MPC algorithms; however, our framework has the important feature that, under Assumption 1, the feasibility is automatically assured by Lemma 2.

III. Numerical examples

In this section, we apply our algorithms to a slightly modified version of the system with "zero-robustness" given by Grimm et al [2] and a chained system [1].

Example 1 Consider the two-dimensional system
$$x_{k+1} = \begin{pmatrix} x_k(1)(1-u_k) \\ \|x_k\|u_k \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w_k, \tag{13}$$
where $u_k \in [0,1]$ is the control input and $w_k \in [0,1]$ is the disturbance. The symbol $x_k(i)$ denotes the i-th component of x_k. For this system, we assume $\|x\| = |x(1)|+|x(2)|$. In fact, the function $f(x) = (x_k(1)(1-u_k), \|x_k\|u_k)^T$ is not C^1 under this norm; nevertheless, our algorithms function effectively.

We apply all our algorithms to (13). For Algorithm 2, we use the cost function $l(i,\phi(i+1,-),u_i) = \|\phi(i+1,-)\|+|u_i|$ (the symbol "$-$" denotes an omission of arguments). Let $x_0 = (a,b)^T$ with $a>0, b>0$, and consider the first block. Because $u_0=1$ and $u_1=0$ gives $x_2=0$ if $w_0=w_1=0$, we begin with $N_0=2$ together with the "reference controller" $(u_0^0,u_1^0)=(1,0)$.

Let $k=0$ in the minor loop of each algorithm. Note that $\phi(1,u_0,w_0,x_0) = (a(1-u_0)+w_0, (a+b)u_0+w_0)^T$ and
$$\phi(2,u_{[0,1]},w_{[0,1]},x_0) = \begin{pmatrix} (a(1-u_0)+w_0)(1-u_1)+w_1 \\ (a+bu_0+2w_0)u_1+w_1 \end{pmatrix}. \tag{14}$$
For Algorithm 1, let
$$h_2(u_{[0,1]},w_{[0,1]}) = \|\phi(2,u_{[0,1]},w_{[0,1]},x_0)\|$$
and
$$\overline{h}_2(u_0,u_1) = \max_{w_{[0,1]}} h_2(u_{[0,1]},w_{[0,1]}).$$
Then, h_2 takes the maximum at $w_0=w_1=1$; therefore,
$$\overline{h}_2(u_0,u_1) = 3+a-au_0+u_1+(a+b)u_0u_1. \tag{15}$$
The minimum of (15) is on the border of $[0,1]\times[0,1]$ because $\partial\overline{h}/\partial u_1 = 1+(a+b)u_0 = 0$ implies $u_0<0$. By evaluating $\overline{h}$ on the border, the minimum of 3 is obtained at $u_0=1$ and $u_1=0$. Hence, $u^{0*}_{[0,1]}=(1,0)$ and $u_0^{0*}=1$ is applied to (13). For Algorithm 2, $J_0^{-1*}=3$ gives the constraint on the inputs, that is, $a-au_0+u_1+(a+b)u_0u_1 \le 0$, which has the unique solution $u^{0*}_{[0,1]}=(1,0)$. Therefore, $u_0^{0*}=1$ is applied to (13). Algorithm 3 is slightly different. For all u_0, $\|\phi(1,u_0,w_0,x_0)\|$ takes the maximum $\overline{h}_1(u_0)=a+bu_0+2$ at $w_0=1$, which in turn takes the minimum $a+2$ at $u_0=0$. Hence, if $a\le1$, $u_0^{0*}=0$ is applied to (13) and the first block terminates; otherwise, the behavior is the same as Algorithm 1.

Next, let $k=1$ (this step does not exist for Algorithm 1 with $a\le1$). For Algorithm 1, we must solve $\min_{u_1^1}\max_{w_1^1} x_1(1)+x_1(2)u_1^1+2w_1^1$, whose solution is $u_1^{1*}=0$ and $w_1^{1*}=1$. Thus, $u_1^{1*}=0$ is applied to (13) and the first block terminates. For Algorithm 2, the constraint becomes
$$w_0+(a+b+w_0)u_1 \le 1. \tag{16}$$
By performing some calculations and omitting constant terms, we obtain the problem $\min_{u_1^1}\max_{w_1^1} 2w_1^1+(1+a+b+w_0)u_1^1$ subject to (16), which has the solution $u_1^{1*}=0$ and $w_1^*=1$. Thus, $u_1^{1*}=0$ is applied to (13) and the first block terminates. For Algorithm 3 with $a\ge1$, the result is the same as that of Algorithm 1.

The analyses of succeeding blocks are the same. For all j, $N_j=2$, and except for Algorithm 3, a two-length control sequence $(u_{M_j},u_{M_j+1})=(1,0)$ is obtained. For Algorithm 3, a one-length control sequence $(u_{M_j})=(0)$ may obtained when $x_j(1)\le1$.

The results of applying each algorithm to (13) with the initial condition $(a,b)=(10,10)$ are shown in Fig. 1, in which the horizontal axis is the time and the vertical axis is the norm of the state at each time instant. For all algorithms, an identical randomly generated disturbance sequence whose components are uniformly distributed in the interval $[0,1]$ is applied. From Fig. 1, it can be observed that all algorithms successfully made the state bounded without causing any instability.

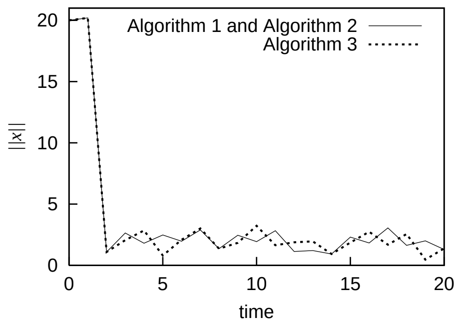

Fig. 1. Results of applying each algorithm to the system with "zero-robustness" .

Example 2 Consider the following four-dimensional chained system [1]:

$$\frac{d}{dt}\begin{pmatrix}x_t(1)\\x_t(2)\\x_t(3)\\x_t(4)\end{pmatrix}=\begin{pmatrix}u_t(1)\\u_t(2)\\x_t(2)u_t(1)\\x_t(3)u_t(1)\end{pmatrix}, \quad (17)$$

where $x_t = (x_t(1), x_t(2), x_t(3), x_t(4))^T$ is the state and $u_t = (u_t(1), u_t(2))^T$ is the input at the time instant t. By assuming that u_t is constant during the sampling period T and integrating, we obtain the following discrete-time system:

$$\begin{pmatrix}x_{k+1}(1)\\x_{k+1}(2)\\x_{k+1}(3)\\x_{k+1}(4)\end{pmatrix}=\begin{pmatrix}x_k(1)\\x_k(2)\\x_k(3)\\x_k(4)\end{pmatrix} + \begin{pmatrix}Tu_k(1)\\Tu_k(2)\\Tu_k(1)x_k(2)+\frac{T^2}{2}u_k(1)u_k(2)\\Tu_k(1)x_k(3)+\frac{T^2}{2}u_k^2(1)x_k(2)+\frac{T^3}{6}u_k^2(1)u_k(2)\end{pmatrix}. \quad (18)$$

In the following, we give the results of applying Algorithm 2, a minimax MPC [9], [10], block MPC [14], and a conventional MPC [7] to (18) with $T = 1$.

To see the robustness, randomly generated disturbance vectors were added to the state at each time instant. For all algorithms, the cost functions of the form (10) were used, with

$$l(i, \phi(i+1, -), u_i) = \|\phi(i+1, -)\|^2 + \|u_i\|^2, i = 0, \ldots, N_j - 1. \quad (19)$$

Note that the block MPC and the conventional MPC assumes zero disturbance in the prediction, whereas Algorithm 2 and the minimax MPC assumes the worst-case disturbances. For Algorithm 2, N_j was initialized with 4, whereas for other algorithms, N_j was fixed at 4. No terminal constraints were used for the minimax MPC, the block MPC and the conventional MPC. The solutions to the minimax problems were obtained approximately by solving the maximization and minimization problems alternatively. Constrained problems were converted to unconstrained ones by the penalty function method.

To solve the optimization problems, Powell's NEWUOA [11] package (Scilab version by Guilbert [5]) was used. All parameters of NEWUOA were tuned so that the slowest algorithm is executable within one second (rhobeg = 10; maxfun 200 and rhoend=10^{-1} for the maximization; maxfun= 400 and rhoend=10^{-2} for the minimization; see the documentation of [5] for the meanings of these parameters). For Algorithm 2, different values of α were used for blocks of different lengths: for blocks with length 1, 2, and 3, $\alpha = 10^{-4}$, whereas for those with length greater than 3, $\alpha = 10^{-1}$. The reason of this possibly unnatural selection is that it is difficult to obtain a good minimizer in limited CPU time for higher dimensional blocks.

The experiments were executed on an IBM-compatible PC (Core2 Quad CPU of 2.40GHz and 2GB of memory) with FreeBSD-7.0-STABLE, Scilab-4.1.2, and Atlas-3.8.0. The CPU time was measured by Scilab's timer() function. The state was initialized with $(10, 10, 10, 10)^T$, and each algorithm was applied over 1,000 steps of time.

The results are shown in Table I, where $E[\cdot]$ denotes the expectation (excluding the initial condition of the state) and τ_k denotes the CPU time required in calculating the control input. In (a), the range of the disturbances was $[-1, 1]$ (uniform distribution), whereas in (b), the range was $[0, 1]$, and the corresponding minimax problems were solved within this range. In this example, all algorithms successfully stabilized the system. As far as $E[\|x_k\|]$ is considered, for (a), the conventional MPC outperformed other algorithms, and Algorithm 2 was next to it, whereas for (b), the minimax MPC outperformed other algorithms, and Algorithm 2 was next to it. Block MPC required the least average CPU time (about 10% of the minimax MPC), but the robustness was relatively poor against disturbances with non-zero expectations. Overall, the behavior of Algorithm 2 and the minimax MPC was similar, but the average CPU time of Algorithm 2 was about 50% $\sim$ 60% of that of the minimax MPC. Algorithm 2 and the minimax MPC required larger inputs (about 150% $\sim$ 290%) of other types.

The distribution of the block lengths of Algorithm 2 is shown in (c). For both (a) and (b), the most frequently used block length was 3. Blocks with length greater than five were never used.

TABLE I
COMPARISON OF SEVERAL MPC ALGORITHMS FOR THE CHAINED SYSTEM.

(a) Disturbance with range $[-1, 1]$.

	$E[\tau_k]$	$E[\|x_k\|]$	$E[\|u_k\|]$
Algorithm 2	0.127234	2.59966	1.85514
Minimax	0.206945	2.75121	2.33358
Block	0.0235859	2.74776	0.797282
Conventional	0.0576406	1.82618	0.837802

(b) Disturbance with range $[0, 1]$.

	$E[\tau_k]$	$E[\|x_k\|]$	$E[\|u_k\|]$
Algorithm 2	0.114047	2.3328	1.31735
Minimax	0.207641	2.20833	1.62888
Block	0.0267187	4.0366	1.09776
Conventional	0.0689844	2.50571	1.13862

(c) The distribution of the block lengths of Algorithm 2 .

Block length	1	2	3	4	5
Case (a)	0	49	97	84	55
Case (b)	0	63	153	68	29

The performance degradation the conventional and block MPC in Table I (b) is due to the bias of the disturbance, and good performance may be recovered by using a disturbance observer.

IV. Conclusion

In this paper, we have constructed robust block model predictive controllers under relatively mild conditions on controllability to the target point. The strategy of these methods is to improve the open-loop controller in a closed-loop fashion by solving minimax problems on-line.

If the target point is an equilibrium and the linear approximation of the system is controllable at the point, a dual-mode scheme that combines one of the proposed methods with linear state feedback around the equilibrium provides a semi-global practical stabilizing controller. To the best of the author's knowledge, this is a rare example of a nonlinear closed-loop stabilizing controller being obtained in a constructive manner merely from the conditions on controllability.

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